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Finite-temperature dynamics of the chaotic maser model

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Abstract. Finite-temperature effects in the chaotic maser model are studied in the framework of a mean field variational approach. Analytic expressions for the relevant thermodynamic properties are given. The chaotic dynamics of the system is extended so as to include finite-temperature effects. Temperature may dramatically influence the behaviour of classical trajectories. A simple physical interpretation is provided.

1. Introduction

The maser model has always been the subject of intense investigation in the various fields of its application. The importance of treating the radiating atoms as a single quantum system was first recognized by Dicke [1], who then correctly described the coherent spontaneous radiation process of the system. An exact solution for the quantized problem, involving N identical two-level atoms interacting with a single mode radiation field, was given by Tavis and Cummings [2], when the dynamics of the system was studied at zero temperature in the rotating wave approximation (RWA), i.e. with inclusion of resonant terms only. As far as thermodynamic properties of the model are concerned, one of the most significant and interesting results was obtained by Hepp and Lieb [3]. They calculated exactly the thermodynamic properties of the system, the free energy in particular, in the thermodynamic limit ($N, V \to \infty$, N/V finite) and showed that for a sufficiently strong coupling between the atoms and the field, the system exhibits a second-order phase transition from the normal phase to superradiance at a certain critical temperature. The results of Hepp and Lieb are rigorous and obtained in the RWA version of the model. Very recently, however, the inclusion of the antiresonant terms of the model have created a lot of interest due to the chaotic nature of its quantum [4] and classical dynamics [5]. In this particular context novel effects connected to the compactness of the atomic Hilbert space (modelled by a spin degree of freedom) have been pointed out [5]. An intriguing question which was still left open was the study of temperature effects on the classical dynamics of the model. The thermodynamical equilibrium properties have been discussed in [6]. It is the aim of the present investigation to shed some light on the question of the finite-temperature dynamics of the model. We first construct the partition function of the model including antiresonant terms in a mean field variational approach. Both degrees of freedom are treated quantum mechanically. Analytic expressions for the

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relevant thermodynamic properties are obtained and the phase transitions are given a very simple interpretation in terms of the photon and atoms statistics. The classical finite-temperature dynamics of the system is then derived and shown to lead to the well known zero-temperature results in this limit. The effect of temperature on the classical chaotic dynamics is discussed and shown to strongly influence the atomic degree of freedom whose phase space is finite.

The paper is structured as follows. Section 2 contains a brief review of the model. In section 3 we study the thermodynamic properties of the model in the context of a mean field variational approach and give a simple interpretation of the phase transition in terms of mean values of photon number and excited atoms. In section 4 the classical finite-temperature dynamics of this chaotic system is obtained and discussed. Section 5 contains some concluding remarks.

2. Dicke model of superradiance

A system consisting of N identical two-level atoms, coupled with an electromagnetic field by means of a dipole interaction, is considered. The system is enclosed in a cavity of volume V. The atoms are kept at fixed position and the dimension V is much smaller than the wavelength of the field so that all atoms see the same field. The quantum Hamiltonian of the model reads

$$H=\varepsilon J_z+\varepsilon a^+a+\frac{G}{N^{1/2}}(J_+a+J_-a^+)+\frac{G'}{N^{1/2}}(J_+a^++J_-a) \tag{2.1}$$

where a^+, a are the Bose operators of a harmonic oscillator mode of frequency ε and J_z and $J_+(J_-)$ are the usual z-component and raising (lowering) spin operators. G and G' are coupling constants. The case with $G'=0$ corresponds to the RWA. In terms of the Pauli spin matrices the Hamiltonian can be rewritten as

$$H=\varepsilon a^+a+\sum_{j=1}^{N}\left[\varepsilon\sigma_j^z+\frac{G}{N^{1/2}}(a\sigma_j^++a^+\sigma_j^-)+\frac{G'}{N^{1/2}}(a^+\sigma_j^++a\sigma_j^-)\right] \tag{2.2}$$

where the index j refers to the jth atom and

$$\sigma_j^+=\sigma_j^x+\mathrm{i}\sigma_j^y \qquad \sigma_j^-=\sigma_j^x-\mathrm{i}\sigma_j^y. \tag{2.3}$$

3. Thermodynamic properties

In this section we construct the free energy of the Hamiltonian (2.2) and study its thermodynamical properties in the framework of a variational mean field calculation. The most general form for the mean field density matrix is given by [7]

$$D_0=K\exp h_{\mathrm{MF}} \tag{3.1}$$

where

$$h_{\mathrm{MF}}=\alpha_1J_z+\alpha_2J_++\alpha_2^*J_-+\beta_1a^+a+\beta_2a^++\beta_2^*a. \tag{3.2}$$

Here K is a normalization constant. The parameters α_1, β_1 are real, the parameters α_2, β_2 are complex. The set of complex parameters $\{\alpha_i, \beta_i\}$ are considered as variational parameters and determined by minimization of the corresponding free energy

$$\beta F=\beta\,\mathrm{Tr}(D_0H)+\mathrm{Tr}(D_0\ln D_0) \tag{3.3}$$

where β is the inverse of the temperature $1/k_B T$, k_B being the Boltzmann constant. The first term on the RHS of (3.3) contains the energy of the system and the last one its entropy.

It is clear that h_{MF} is the mean field Hamiltonian except for a multiplicative factor. It is convenient to consider the unitary transformation which diagonalizes D_0:

$$U=\exp[\mathrm{i}(\eta J_+ + \eta^* J_- + \xi a^+ + \xi^* a)] \tag{3.4}$$

(with η, η^*, ξ, ξ^* complex numbers),

$$D = U D_0 U^+ \tag{3.5}$$

with

$$D = K \exp(\gamma J_z + \gamma' a^+ a). \tag{3.6}$$

In this case (3.3) can be rewritten as

$$\begin{aligned}\beta F &= \beta \operatorname{Tr}(U^+ D U H) + \operatorname{Tr}(D \ln D)\\ &= \beta \operatorname{Tr}(D U H U^+) + \operatorname{Tr}(D \ln D).\end{aligned} \tag{3.7}$$

Now the free energy becomes a function of the temperature, of the parameters γ and γ' and the parameters of the unitary transformation η, η^*, ξ and ξ^*. It is easy to check that for the static properties of the system it is enough to consider

$$\eta = \frac{\theta}{2\mathrm{i}} \qquad \theta \text{ a real number} \tag{3.8a}$$

and

$$\xi = \xi^*. \tag{3.8b}$$

To study the dynamical equations of motion, in the next section, we shall need the most general transformation of the form (3.4). The free energy of the system is easily computed. The result is

$$\begin{aligned}\beta F = \beta\left[\varepsilon \frac{N}{2}\cos\theta \tanh\frac{\gamma}{2} + \frac{\varepsilon \mathrm{e}^{\gamma'}}{1-\mathrm{e}^{\gamma'}} + \varepsilon\xi^2 + N^{1/2}\xi(G+G')\sin\theta\tanh\frac{\gamma}{2}\right]\\ + \frac{N}{2}\gamma\tanh\frac{\gamma}{2} + \gamma'\frac{\mathrm{e}^{\gamma'}}{1-\mathrm{e}^{\gamma'}} + \ln(1-\mathrm{e}^{\gamma'}) - N\ln\left(2\cosh\frac{\gamma}{2}\right).\end{aligned} \tag{3.9}$$

One can now determine the parameters γ, γ', θ and ξ by minimizing the above expression with respect to these variables. We get for the variations with respect to θ and ξ

$$\begin{aligned}\frac{\partial\beta F}{\partial\xi} &= 0 = 2\varepsilon\xi + N^{1/2}(G+G')\sin\theta\tanh\frac{\gamma}{2}\\ \frac{\partial\beta F}{\partial\theta} &= 0 = -\varepsilon\frac{N}{2}\sin\theta\tanh\frac{\gamma}{2} - \frac{(G+G')^2}{\varepsilon}\frac{N}{2}\sin\theta\cos\theta\tanh^2\frac{\gamma}{2}.\end{aligned} \tag{3.10}$$

These equations have two solutions, which correspond to the normal and to the superradiant phase, respectively,

$$\theta = 0 \qquad \xi = 0 \qquad \text{(normal phase)} \tag{3.11a}$$

and

$$\cos\theta = -\frac{\varepsilon^2}{(G+G')^2}\frac{1}{\tanh\gamma/2}$$

$$\xi = -\frac{N^{1/2}}{2}\frac{G+G'}{\varepsilon}\sin\theta\tanh\gamma/2 \qquad \text{(superradiant phase).} \tag{3.11b}$$

Notice that the solution (3.11b) is only possible for sufficiently strong coupling, i.e. $(G+G')^2 > \varepsilon^2$. The parameters γ, γ' obtained in the same manner are of course related to the temperature of the system and are given by

$$\gamma = -\beta\varepsilon \qquad \text{(normal phase)} \tag{3.12a}$$

$$\gamma = -\beta\varepsilon/\cos\theta \qquad \text{(superradiant phase)} \tag{3.12b}$$

$$\gamma' = -\beta\varepsilon \qquad \text{(both phases).} \tag{3.12c}$$

We see from the above equations that the existence of the superradiant phase depends on the temperature. For $(G+G')^2 < \varepsilon^2$ no phase transition occurs in the system at any temperature. For $(G+G')^2 > \varepsilon^2$ there is a critical temperature T_c given by the relation (for its inverse β_c)

$$\frac{\varepsilon^2}{(G+G')^2} = -\tanh\varepsilon\beta_c/2 \tag{3.13}$$

at which the system changes discontinuously from one state to the other. Equation (3.13) has been obtained in [8] by a different procedure and is equivalent to the mathematically rigorous result of Hepp and Lieb [3] in the case $G'=0$.

The average density of photons and of excited atoms in both phases can be easily computed

$$\frac{\langle a^+a\rangle}{N} = -\frac{1}{N}\mathrm{Tr}(D_0a^+a) = -\frac{1}{N}\mathrm{Tr}(DUa^+aU^+)$$

$$= \begin{cases} 0 & \text{normal phase} \quad (3.14a) \\ (G+G')/\varepsilon\,\sin\theta\tanh\gamma/2 & \text{superradient phase} \quad (3.14b) \end{cases}$$

where β in the superradiant phase is determined by (3.12b),

$$\gamma = \frac{\beta(G+G')^2}{\varepsilon}\tanh\gamma/2. \tag{3.15}$$

The solutions to this equation can be obtained graphically or numerically and results are shown in figure 1 for the indicated parameter values.

The average density of excited atoms is given by

$$\frac{\langle J_z\rangle}{N} = \frac{1}{N}\mathrm{Tr}(D_0J_z) = \frac{1}{N}\mathrm{Tr}(DUJ_zU^+)$$

$$= \tfrac{1}{2}\tanh\gamma/2\begin{cases} \gamma = -\beta\varepsilon & \text{normal phase} \quad (3.16a) \\ \gamma(\beta/\varepsilon)(G+G')^2\tanh\tfrac{1}{2}\gamma & \text{superradiant phase} \quad (3.16b) \end{cases}$$

in both phases.

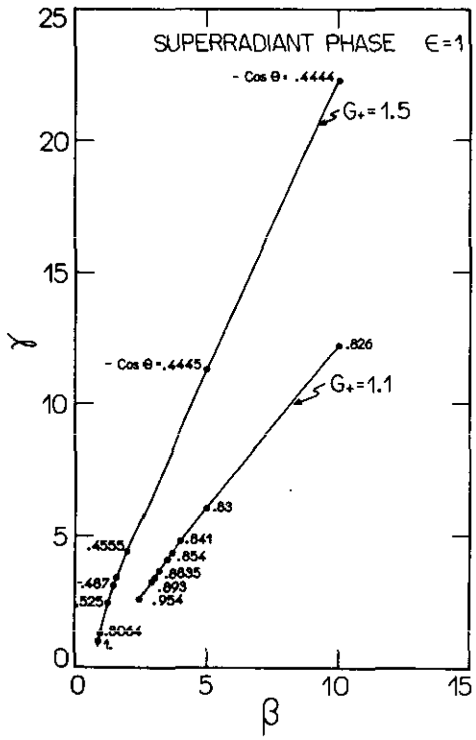


Figure 1. The relation between the variational parameter γ and the inverse temperature β of the system in the superradiant phase for two different values of the photon–atom coupling parameter G_+. The dots represent the values of the parameter $\cos\theta$ which relates β to γ.

The set of equations (3.14*a*), (3.14*b*) and (3.16) provides us with a very simple physical picture of what is going on. For the normal phase, the average photon density is always zero and the average density of excited atoms is different from zero and fixed by (3.16*a*). In contrast, in the superradiant phase the average photon density is different from zero but decreases as a function of temperature until the critical temperature is reached. At this point the average photon density is zero and the average density of excited atoms tends to its normal phase value. Here, the superradiant phase merges into the normal phase. This is illustrated in figure 2 for the indicated values of the coupling parameters. Figures 3(*a*) and 3(*b*) show the behaviour of the energy and entropy in the normal and superradiant phases as functions of the temperature, under the same conditions.

It is also easy to check that while the superradiant phase exists, its free energy is a minimum and the corresponding free energy of the normal phase a maximum. Other than this, the normal phase has always the minimum energy.

In order to get a quantitative idea of how reasonable such an approximation is, we compare the result obtained here for the energy at $T=0$ with the exact one for parameter values $G=1$, $G'=0.4$: $E_{ex}=-5.559\,43$, $E_{MF}=-5.556$.

4. Classical finite-temperature dynamics

The following procedure to derive the classical finite-temperature dynamics of systems is very general and should be valid for all systems described in terms of generators of

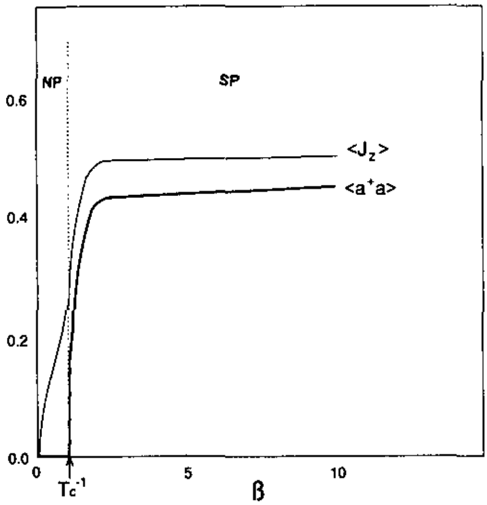


Figure 2. The average density of excited atoms $\langle J_z \rangle$ as well as the average photon density $\langle a^+ a \rangle$ of the system as functions of the inverse temperature β in the superradiant phase (SP; $\beta > \beta_c$) and in the normal phase (NP, $\beta < \beta_c$) for coupling constants $G_+ = 1.5$ and $G = 0.5$ and $\varepsilon = 1$. The value of the inverse of the critical temperature T_c^{-1} at which the system changes phase is indicated by an arrow and the two phases separated by a vertical dotted line.

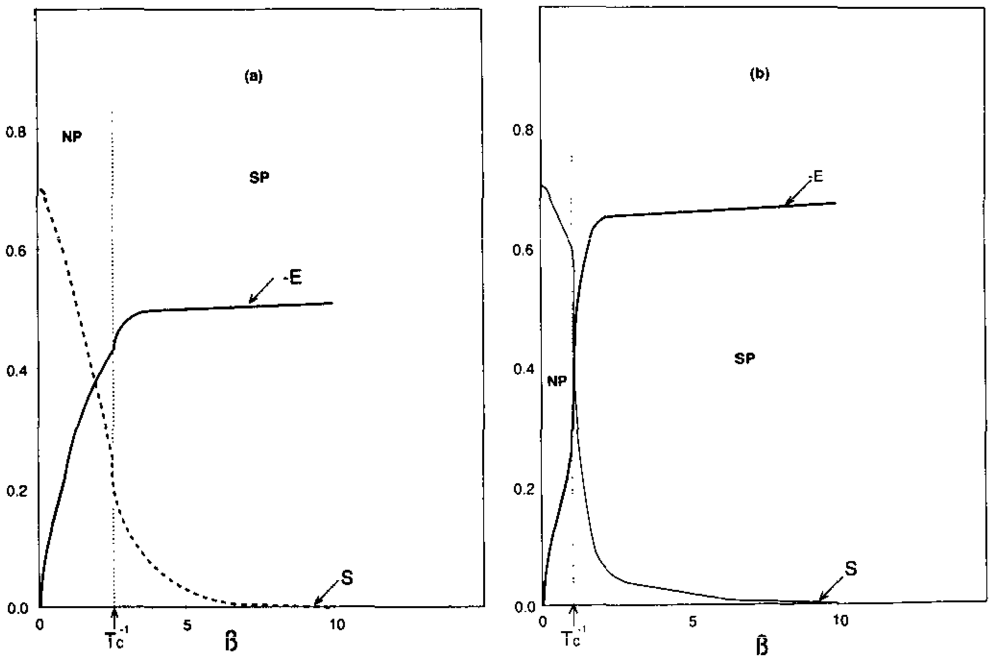


Figure 3. (a) The energy E and entropy S in the normal (NP) and superradiant (SP) phase as functions of the inverse temperature β for $G_+ = 1.5$ and $G_- = 0.5$, $\varepsilon = 1$. (b) The same as (a) for $G_+ = 1.1$ and $G\_0.9$.

Lie groups. The maser model consists of one possible application of the method [9]. It consists of constructing appropriate pairs of canonical variables and calculating the corresponding classical Lagrangian of the system ($\hbar=1$),

$$L=\mathrm{i}\,\mathrm{Tr}(DU\dot{U}^{+})-\mathrm{Tr}(DUHU^{+}) \tag{4.1}$$

with U in the form given in the previous section, (3.4),

$$U=\exp[\mathrm{i}(\eta J_{+}+\eta^{*}J_{-}+\xi a^{+}+\xi^{*}a)].$$

After some algebra one obtains the following intermediate results, necessary to construct the Lagrangian,

$$\mathrm{Tr}(DU\dot{U}^{+})=-\bar{J}_z(\eta\dot{\eta}^{*}-\dot{\eta}\eta^{*})\sin^2\sqrt{\eta\eta^{*}} \tag{4.2a}$$

$$\mathrm{Tr}(DUJ_{-}U^{+})=\mathrm{i}\bar{J}_z\frac{\eta}{\sqrt{\eta\eta^{*}}}\sin 2\sqrt{\eta\eta^{*}} \tag{4.2b}$$

$$\mathrm{Tr}(DUJ_zU^{+})=\bar{J}_z\cos 2\sqrt{\eta\eta^{*}} \tag{4.2c}$$

$$\mathrm{Tr}(DUa^{+}aU^{+})=\xi\xi^{*} \tag{4.2d}$$

$$\mathrm{Tr}(DUaU^{+})=-\xi \tag{4.2e}$$

$$\mathrm{Tr}(DUa^{+}U^{+})=-\xi^{*} \tag{4.2f}$$

with $\bar{J}_z=\mathrm{Tr}(D_0J_z)$. We immediately notice that the two variables η, η^{*} in the unitary transformation are not canonically conjugate. But it is possible to find canonical variables if one performs the transformation [10]

$$\delta=\eta(-2\bar{J}_z)^{1/2}\frac{1}{\sqrt{\eta\eta^{*}}}\sin\sqrt{\eta\eta^{*}} \tag{4.3a}$$

$$\delta^{*}=\eta^{*}(-2\bar{J}_z)^{1/2}\frac{1}{\sqrt{\eta\eta^{*}}}\sin\sqrt{\eta\eta^{*}}. \tag{4.3b}$$

Finally we get

$$\begin{aligned}L=&\frac{\mathrm{i}}{2}(\dot{\delta}\delta^{*}-\dot{\delta}^{*}\delta)+\frac{\mathrm{i}}{2}(\dot{\xi}\xi^{*}-\dot{\xi}^{*}\xi)-\varepsilon\xi^{*}\xi+\varepsilon\delta\delta^{*}+\varepsilon\bar{J}_z\\&+\frac{G}{N^{1/2}}(\xi\delta^{*}-\xi^{*}\delta)(-\mathrm{i})\sqrt{-2\bar{J}_z}\sqrt{1-\frac{\delta\delta^{*}}{-2\bar{J}_z}}\\&+\frac{G'}{N^{1/2}}(\xi^{*}\delta^{*}-\xi\delta)(-\mathrm{i})\sqrt{-2\bar{J}_z}\sqrt{1-\frac{\delta\delta^{*}}{-2\bar{J}_z}}.\end{aligned} \tag{4.4}$$

Defining now new variables

$$\mathrm{i}\eta=\frac{p_1+\mathrm{i}q_1}{2^{1/2}} \tag{4.5a}$$

$$\xi=\frac{p_2+\mathrm{i}q_2}{2^{1/2}} \tag{4.5b}$$

we obtain the classical Lagrangian which corresponds to the quantum (chaotic) maser system with finite temperature,

$$L=p_1\dot{q}_1+p_2\dot{q}_2-\varepsilon H_1-\varepsilon H_2-\varepsilon\bar{J}_z-\left[\frac{-2\bar{J}_z-H_1}{N}\right]^{1/2}(G_{+}p_1p_2+G_{-}q_1q_2) \tag{4.6}$$

with $H_1=\frac{1}{2}(p_1^2+q_1^2)$, $H_2=\frac{1}{2}(p_2^2+q_2^2)$, $G_\pm=G\pm G'$, from which the classical equations of motion can be derived:

$$\begin{aligned}
\dot{q}_1&=\varepsilon p_1+G_+\left[\frac{-2\bar{J}_z-H_1}{N}\right]^{1/2}p_2-\frac{p_1}{2[N(-2\bar{J}_z-H_1)]^{1/2}}(G_+p_1p_2+G_-q_1q_2)\\
\dot{p}_1&=-\varepsilon q_1-G_-\left[\frac{-2\bar{J}_z-H_1}{N}\right]^{1/2}q_2+\frac{q_1}{2[N(-2\bar{J}_z-H_1)]^{1/2}}(G_+p_1p_2+G_-q_1q_2)\\
\dot{q}_2&=\varepsilon p_2+G_+\left[\frac{-2\bar{J}_z-H_1}{N}\right]^{1/2}p_1\\
\dot{p}_2&=-\varepsilon q_2-G_-\left[\frac{-2\bar{J}_z-H_1}{N}\right]^{1/2}q_1.
\end{aligned}\tag{4.7}$$

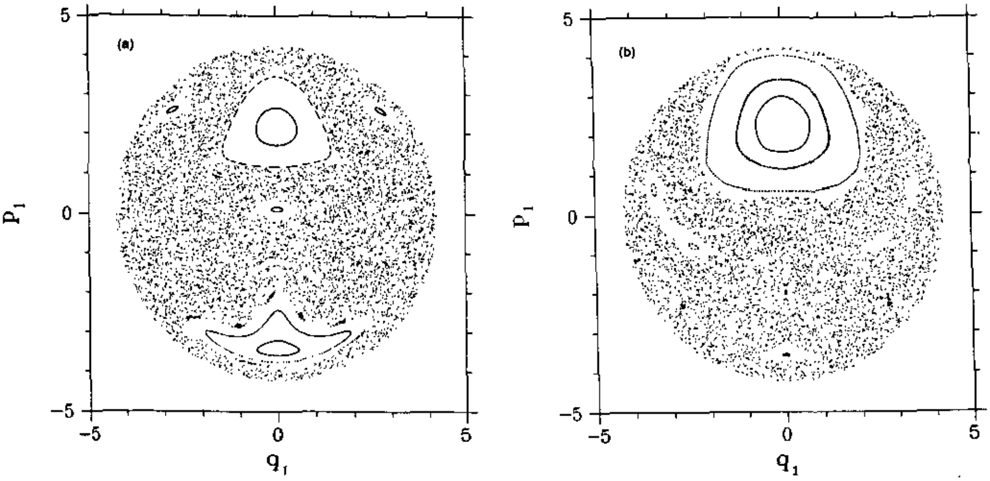
The above equations have been derived in the limit of zero temperature [6, 8] and shown to exhibit classically chaotic behaviour. Our equations reduce precisely to the ones in [6, 8] in the zero-temperature limit where $\bar{J}_z=-N/2$.

The only significant difference from the equations with temperature resides in the terms involving the square root. As extensively discussed in [6] and [7] the physics of this term stems from the finiteness of the atomic (or spin) phase space. The maximum amount of energy allowed for this degree of freedom is $-2\bar{J}_z$. This has a dramatic effect on periodic orbits of sufficiently high energy. They exhibit many crossings and cusps, since they cannot escape from the border of their phase space, $H_1=N$. The temperature will enhance this effect in the sense that it will become apparent for smaller energies. For finite temperatures the 'border' of this phase space shrinks, $-2\bar{J}_z<N$, and its physical effects—the many self-crossings and cusps of periodic orbits—will appear sooner, for lower energies, as in the corresponding zero-temperature limit. A simple physical interpretation can be given. When the system is at zero temperature, all atoms are in their ground states (if we take for the simplicity the normal phase) and therefore, as the total energy of the system increases, more and more atoms are being excited until all of them are excited and no more energy can be pumped into the spin degree of freedom. For finite temperatures, the 'ground state' of the system contains already some excited atoms, so that the total energy still allowed is smaller. What one sees here is the classical analogue of such physics.

To illustrate the phase space occupied by chaotic orbits we show in figures 4(a), 4(b) and 4(c) Poincaré sections for the spin degree of freedom (q_1, p_1) at $q_2=0$, for the three different values of the total energy $E=6.2$, 8.5 and 25, respectively. Note that there exists a scaling which lets us interpret these figures in different ways. This can be seen if one rewrites the Lagrangian as

$$L=p_1\dot{q}_1+p_2\dot{q}_2-\varepsilon H_1-\varepsilon H_2-\varepsilon\bar{J}_z-\left[\frac{-2\bar{J}_z-H_1}{-2\bar{J}_z}\right]^{1/2}(\tilde{G}_+p_1p_2+\tilde{G}_-q_1q_2)\tag{4.8}$$

where $\tilde{G}_\pm=(-2\bar{J}_z/N)^{1/2}$, $G_\pm=(-\tanh\frac{1}{2}\gamma)\,G_\pm$, with γ a function of β, see (3.16). That is, a given solution of the equations of motion can be obtained by any combination of the parameters N, β and $G_\pm$, as long as ε, $\tilde{G}_\pm$ and $\bar{J}_z$ have certain fixed values. As an example, figures 4(a), 4(b) and 4(c) can each be regarded as representing a system in the normal phase with alternatively $N=9$ at inverse temperature $\beta=\infty$, or $N=16$ at $\beta=1.27$, or $N=30$ at $\beta=0.62$, or $N=50$ at $\beta=0.36$, etc, with the coupling constants



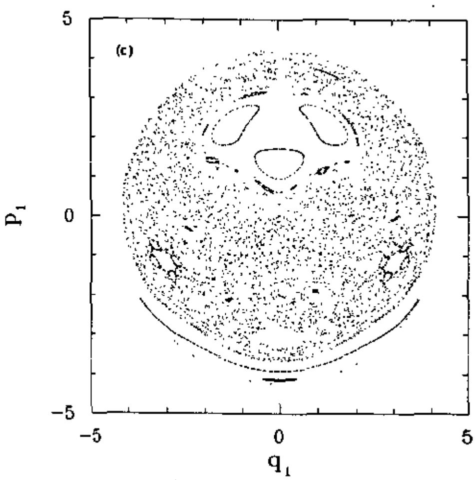


Figure 4. Poincaré sections of the spin degree of freedom at $q_2=0$ for $\bar{J}_z=-\frac{9}{2}$, $\varepsilon=1$ and $\tilde{G}_+=0.7$, $\tilde{G}_-=0.3$. These figures can be interpreted in terms of different temperatures and particle numbers, see the discussion in the text. The values of the total energy of the system are (a) $E=6.2$, (b) 8.5 and (c) 25.

$G_\pm$ adjusted correspondingly. The values of the other parameters are $\bar{J}_z=-9/2$, $\tilde{G}_+=0.7$, $\tilde{G}_-=0.3$ and $\varepsilon=1$.

For system energies much smaller than the one in figure 4(a), $E \ll 6$, there is little chaos. For large energies the chaotic behaviour diminishes again since the dynamics of the oscillator term dominates. This is because the phase space of the spin degree of freedom is restricted and the phase space of the oscillator degree of freedom is not. In the range of energies presented in figures 4(a) and 4(b) the amount of chaos is large. It starts to get smaller in figure 4(c).

5. Conclusions

We have studied finite-temperature effects in the chaotic maser model in the framework of a mean field variational approach. Analytic expressions for the relevant thermodynamic properties have been derived and a simple picture for the phase transition emerges. The method is then extended to describe large amplitude, nonlinear dynamics of the system at finite temperatures. The effect of the temperature on classical periodic orbits is obtained and a physical interpretation is given.

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